Astron. & Astrophys. 9, 472—474 (1970)

Research Note

Newtonian Electrodynamics from General-Relativistic Arguments

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Received July 23, 1970

In general relativity, a vacuum is electrically polarized in the presence of a magnetic field whenever the mixed components of the metric tensor do not vanish (which occurs in astronomy around a massive rotating object because of the dragging of the inertial frame). This note illustrates the phenomenon by making use of the fact that it does not require curvature and, hence, may occur in a *flat* space-time: In fact, in corotating coordinates the vacuum around a rotating, magnetic, newtonian star is electrically charged. By assuming near sphericity of the star and an external 2^n-pole poloidal magnetic field, it is easy to evaluate the electrostatic potential, which reflects the existence of the fictitious charge density with a non-harmonic term proportional to $(1/r)^n P_{n+1}(\cos\theta)$. This note also establishes a convenient formalism for the treatment of the general-relativistic aspects of the same problem.

Key words: general relativity — electromagnetism

Schiff (1939) recognized that the very formulation of electrodynamics in rotating coordinates is a general-relativistic problem, even when the Riemann tensor vanishes (a similar conclusion can be inferred from Landau and Lifshitz 1962, p. 296). Although no need for such a formalism arises in the galilean space-time to be considered below and special-relativistic considerations suffice (see Goldreich and Julian, 1969 and the Appendix), the study of the fictitious charge (and current) densities arising with the adoption of rotating coordinates, in analogy to the Coriolis and centrifugal forces of mechanics, illustrates the nature of the virtual charges in a curved space (Occhionero, 1970).

Irvine (1964) clarifies the controversial problem of relating, in any metric $g_{\alpha\beta}$, the em tensor $F_{\alpha\beta}$ to the physical components of the em field, as actually measured by any specific observer: The ambiguities often met in the literature can be avoided only by using the observer's orthonormal tetrad, OT (Synge, 1960), which translates in mathematical terms the notion of a locally tangent inertial system of reference (here the locally corotating observer). With $x^0 = ct$ and space coordinates x^1, x^2 and x^3, the OT, $\lambda^\alpha_{(\mu)}$, defines uniquely its observer's physical em fields as follows:

$$F_{(\mu)\,(\nu)} = \lambda^\alpha_{(\mu)}\lambda^\beta_{(\nu)}F_{\alpha\beta}\,,$$
$$E_{(k)} = F_{(k)\,(0)}\,, \tag{1}$$
$$B_{(1)} = F_{(2)\,(3)} \text{ and cyclic}\,,$$

where Greek letters range from 0 to 3 and Latin from 1 to 3.

The poloidal, axially symmetric, stationary magnetic field of a rotating or non-rotating, newtonian or relativistic star is best studied in spherical polar coordinates,

$$x^1 = r,\ x^2 = \theta,\ x^3 = \phi\,, \tag{2}$$

from the generalized wave-equations for the vector potential, A_α,

$$F_{\alpha\beta} = A_{\alpha,\beta} - A_{\beta,\alpha}\,. \tag{3}$$

Under time- and azimuth-independence, an obvious gauge is satisfied and the equations for A_0 and A_3 may be cast in the convenient form

$$\Box A_W + 2\sum_{N=1,2} g^{NN}\left\{\begin{matrix}W\\WN\end{matrix}\right\}F_{NW} = 2\sum_{N=1,2} g^{NN}\left\{\begin{matrix}Z\\WN\end{matrix}\right\}$$
$$\cdot F_{ZN} + (4\pi/c)j_W \qquad W, Z = 0{,}3\,,\ W \neq Z\,, \tag{4}$$
$$\Box = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\beta}\sqrt{-g}\,g^{\beta\gamma}\frac{\partial}{\partial x^\gamma}\,.$$

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For a non-rotating, spherical and newtonian star,

$$g_{\alpha\beta} = \mathrm{diag}\,(+1, -1, -r^2, -r^2\sin^2\theta)\,, \quad (5)$$

$$-A_{3,rr} - (1/r^2)\Lambda\,(A_3) = (4\pi/c)j_3\,, \quad (6)$$

$$\Lambda = \sin\theta\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\,. \quad (7)$$

The Λ-operator (which is not self-adjoint) has the following eigenfunctions (which are not orthogonal)

$$\Lambda\psi_n(\theta) + n(n+1)\psi_n(\theta) = 0\,,$$

$$\psi_n(\theta) = \frac{1}{n(n+1)}\sin\theta\frac{d}{d\theta}P_n(\cos\theta) \quad (8)$$

$$= \frac{1}{(2n+1)}[P_{n+1} - P_{n-1}]\,.$$

They satisfy the relations

$$\cos\theta P_n + n\psi_n = P_{n+1}\,,$$
$$\cos\theta P_n - (n+1)\psi_n = P_{n-1}\,, \quad (9)$$

and can be used to describe the 2^n-pole magnetic field. The latter must be continuous at the surface of the star and regular at its center: A current layer at $r = L < R$ satisfies these requirements in a schematic, but convenient way, since it makes Eq. (6) actually homogeneous and provides model-independent considerations (L disappears from the final results). The appropriate continuous solution of (6) is, then,

$$\text{for } 0 \leqq r \leqq L,\ A_3 = BR^2(R/L)^{2n+1}(r/R)^{n+1}\psi_n(\theta)\,,$$
$$\text{for } L \leqq r,\ A_3 = BR^2(R/r)^n\psi_n(\theta)\,, \quad (10)$$

where the chosen normalization makes equal to B the field on the poles, $r = R$, $\theta = 0$ and π. The OT of the observer at rest with respect to the coordinates is

$$\lambda^{\alpha}{}_{(0)} = (1, 0, 0, 0,),\ \lambda^{\alpha}{}_{(1)} = (0, 1, 0, 0)\,, \quad (11)$$
$$\lambda^{\alpha}{}_{(2)} = (0, 0, 1/r, 0),\ \lambda^{\alpha}{}_{(3)} = (0, 0, 0, 1/r\sin\theta)\,,$$

and operates in (1) as a set of geometrical scaling functions:

$$B_{(r)} = B(R/L)^{2n+1}(r/R)^{n-1}P_n\,,$$

for $0 \leqq r < L$, (12)

$$B_{(\theta)} = B(R/L)^{2n+1}(r/R)^{n-1}\frac{1}{n}\frac{dP_n}{d\theta}\,,$$

$$B_{(r)} = B(R/r)^{n+2}P_n\,,$$

for $L < r$, (13)

$$B_{(\theta)} = -B(R/r)^{n+2}\frac{1}{n+1}\frac{dP_n}{d\theta}\,.$$

Let now the star be imagined in slow rotation with uniform angular velocity Ω around the $\theta = 0$ axis, and let corotating coordinates be chosen

$$t' = t,\ r' = r,\ \theta' = \theta,\ \phi' = \phi - \Omega t\,. \quad (14)$$

By neglecting, for shortness, all the Ω^2-corrections and, consistently, the rotational deformation of the star, the $g_{\alpha\beta}$'s retain the form (5) on the diagonal, but acquire a new off-diagonal element,

$$g'_{03} = -(\Omega/c)r^2\sin^2\theta\,, \quad (15)$$

which, in this formulation, couples electric and magnetic fields (Landau and Lifshitz, 1962; Occhionero, 1970). Primes will be omitted henceforth. The OT of the observer locally at rest with respect to these coordinates is

$$\mu^{\alpha}_{(0)} = \lambda^{\alpha}_{(0)},\ \mu^{\alpha}_{(1)} = \lambda^{\alpha}_{(1)},\ \mu^{\alpha}_{(2)} = \lambda^{\alpha}_{(2)}\,, \quad (16)$$
$$\mu^{\alpha}_{(3)} = \left(\frac{\Omega}{c}r\sin\theta, 0, 0, \frac{1}{r\sin\theta}\right),$$

the λ's being defined in (11). For the observer (16), the relation of the physical magnetic field to $F_{\alpha\beta}$ is the same as for (11) (aside from Ω^2-corrections which imply a tangential discontinuity of B on the star surface and a fictitious surface current) and the equation for A_3, (6), remains unchanged: Hence the magnetic field is given again by (12) and (13). The electric field is

$$E_{(r)} = F_{10},\ E_{(\theta)} = F_{20}/r,\ E_{(\phi)} = F_{30}/r\sin\theta = 0\,, \quad (17)$$

so that its vanishing inside the conductor is equivalent to the constancy of A_0 therein. For $W = 0$, Eq. (14) yields

$$-\nabla^2 A_0 = -\frac{2\Omega}{cr^2}(rF_{31} + \cot\theta F_{32}) + 4\pi\varrho\,, \quad (18)$$

the r.h.s. of which consists of the fictitious charge density (Schiff, 1939) and of the real polarization ϱ; only the latter vanishes outside the star under the present assumption of perfect vacuum. Inside, the necessary condition for the constancy of A_0, is the vanishing of the r.h.s. of (18): Hence,

$$\varrho = -\frac{\Omega B}{2\pi c}(R/L)^{2n+1}(r/R)^{n-1}P_{n-1}(\cos\theta),$$
$$\text{for } 0 \leqq r < L\,,$$

$$\varrho = -\frac{\Omega B}{2\pi c}(R/r)^{n+2}P_{n+1}(\cos\theta),\ \text{for } L < r < R\,. \quad (19)$$

The volume integral of ϱ over the star vanishes for any $n > 1$, but for $n = 1$, there is a net charge inside the star,

$$Q = -\tfrac{2}{3}(\Omega BR^3/c)\,, \quad (20)$$

even if the total charge of the star may still be zero, as in Goldreich and Julian (1969). For $r \geqq R$ (with $\varrho = 0$), the solution to (18) is the sum of an inhomogeneous integral and a harmonic function; the arbitrary constants are determined by the continuity of A_0:

$$A_0 \equiv 0, \text{ for } r \leqq R\,, \tag{21}$$

$$A_0 = \frac{\Omega B R^2}{(2n+1)\,c}\,[(R/r)^n - (R/r)^{n+2}]\,P_{n+1}(\cos\theta), \quad \text{for } r \geqq R\,,$$

which proves the statement of the abstract. Finally, the surface charge density is

$$\sigma = -\frac{\Omega B R}{2\pi c\,(2n+1)}\,P_{n+1}(\cos\theta)\,, \tag{22}$$

and integrates to zero for all n's.

Appendix

Special relativistic considerations provide an alternative evaluation of (21): The OT of the fixed observer is $\lambda^{\alpha}_{(\mu)}$ of (11), and the OT of the locally corotating observer, as viewed from the non-rotating coordinates, is

$$\nu^{\alpha}_{(0)} = (1, 0, 0, \Omega/c), \; \nu^{\alpha}_{(k)} = \mu^{\alpha}_{(k)}\,, \tag{A1}$$

with the μ's of (16). The relation between the em fields measured by the two observers can be worked out from (1) (Irvine, 1964) and is, of course, a Lorentz transformation (exact when the γ's are inserted),

$$\boldsymbol{E}(\nu) = \boldsymbol{E}(\lambda) + 1/c\,(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{B}(\lambda)\,, \tag{A2}$$

$$\boldsymbol{B}(\nu) = \boldsymbol{B}(\lambda)\,. \tag{A2}$$

Internally, $\boldsymbol{E}(\nu) \equiv 0$ (Goldreich and Julian, 1969), and the expressions (19) and (20) are immediately derived from

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = -2\boldsymbol{\Omega} \cdot \boldsymbol{B} + (\boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \boldsymbol{\nabla} \times \boldsymbol{B}\,,$$

Externally, $\boldsymbol{E}(\lambda)$ is the gradient of a harmonic function, but $\boldsymbol{E}(\nu)$ is not, because of (A2):

$$\boldsymbol{E}(\lambda) = \sum_{s=0}^{\infty} K_s \boldsymbol{\nabla}[(R/r)^{s+1} P_s(\cos\theta)]\,, \tag{A3}$$

$$1/c\,(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{B} = \frac{\Omega B R^2}{(2n+1)\,c}\boldsymbol{\nabla}[(R/r)^n (P_{n-1} - P_{n+1})]\,. \tag{A4}$$

The constants in (A3) ($K_0 = 0$ in Goldreich and Julian, 1969) are determined from

$$E_{(\theta)}(\nu) = 0 \text{ at } r = R\,. \tag{A5}$$

This gives

$$K_s = 0 \text{ for } s \neq n-1, n+1\,,$$

$$K_{n+1} = -K_{n-1} = \frac{\Omega B R^2}{(2n+1)\,c},$$

$$\boldsymbol{E}(\nu) = -\boldsymbol{\nabla}\Phi\,,$$

$$\Phi = \frac{\Omega B R^2}{(2n+1)\,c}\,[(R/r)^n - (R/r)^{n+2}]\,P_{n+1}(\cos\theta)\,,$$

in agreement with (21).

Acknowledgements. I am deeply indebted to Dr. P. Goldreich for illuminating discussions and to Dr. R. Jastrow for hospitality at the Institute for Space Studies.

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